Problem 1.23

(For masochists only.) Prove product rules (ii) and (vi). Refer to Prob. 1.22 for the definition of $(\mathbf{A}\cdot\nabla)\mathbf{B}$.

Solution

Proof of (ii)

The aim is to prove that

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}. \tag{ii}$$

Write out the first two terms on the right side explicitly.

$$\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right)\right] \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial}{\partial x_j}B_k\right] \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial}{\partial x_j}A_k\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial B_k}{\partial x_j}\right) \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}B_i\frac{\partial A_k}{\partial x_j}
\end{aligned}$$

Continue the simplification.

$$
\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})A_i\frac{\partial B_k}{\partial x_j} \\
&\quad+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}A_i\frac{\partial B_k}{\partial x_j}-\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}B_i\frac{\partial A_k}{\partial x_j}-\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{ik}A_i\frac{\partial B_k}{\partial x_j}-\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\delta_{ij}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{ik}B_i\frac{\partial A_k}{\partial x_j}-\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\delta_{ij}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_jA_k\frac{\partial B_k}{\partial x_j}-\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_kA_j\frac{\partial B_k}{\partial x_j} \\
&\quad+\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_jB_k\frac{\partial A_k}{\partial x_j}-\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_kB_j\frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\left(A_k\frac{\partial B_k}{\partial x_j}+B_k\frac{\partial A_k}{\partial x_j}\right) \\
&\quad-\sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right)-\sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_kB_k)-\sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right)-\sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}A_kB_k\right)-\sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right)-\sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \nabla(\mathbf{A}\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\mathbf{B}-(\mathbf{B}\cdot\nabla)\mathbf{A}
\end{aligned}
$$

Therefore,

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}.$$

Proof of (iii)

The aim is to prove that

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f). \tag{iii}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \cdot (f\mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[f\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j f\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} A_j f \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \left(\frac{\partial A_j}{\partial x_i} f + A_j \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i} f + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_j \frac{\partial f}{\partial x_i} \\
&= f\left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i}\right] + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_i \frac{\partial f}{\partial x_j} \\
&= f\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] + \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial f}{\partial x_j}\right) \\
&= f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f
\end{aligned}$$

Proof of (vi)

The aim is to prove that

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}). \tag{vi}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) A_j B_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} A_j B_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_k} A_j B_k - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_j} A_j B_k \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \left(\frac{\partial A_j}{\partial x_k} B_k + A_j \frac{\partial B_k}{\partial x_k}\right) - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\frac{\partial A_j}{\partial x_j} B_k + A_j \frac{\partial B_k}{\partial x_j}\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial A_j}{\partial x_k} B_k + \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j A_j \frac{\partial B_k}{\partial x_k} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial A_j}{\partial x_j} B_k - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k A_j \frac{\partial B_k}{\partial x_j} \\
&= \sum_{k=1}^{3} B_k \frac{\partial}{\partial x_k}\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) + \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \sum_{k=1}^{3} \frac{\partial B_k}{\partial x_k} - \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right) \sum_{j=1}^{3} \frac{\partial A_j}{\partial x_j} - \sum_{j=1}^{3} A_j \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right) \\
&= (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}
\end{aligned}$$